4. FREQUENCY RESPONSE ANALYSIS

4.1 Correlation between time response and frequency response

Frequency-Domain Specifications :-

The following frequency-domain specifications are often used :

(a) Resonant Peak (M_r) :

The resonant peak M_r is the maximum value of $|M(j\omega)|$.

(b) Resonant Frequency (ω_r) :

The resonant frequency ω_r is the frequency at which the peak resonance M_r occurs.

(c) Bandwidth (BW) :

The bandwidth (BW) is the frequency at which $|M(j\omega)|$ drops to 70.7% of, or 3 dB down from, its zero frequency value.

M_r, ω_r and BW of the prototype second order System

Consider the closed-loop transfer function (CLTF) of the prototype second order system.

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{—} (1)$$

At sinusoidal steady state $s = j\omega$; eqn. (1) becomes

$$M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} \quad \text{—} (2)$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\xi\omega\omega_n + \omega_n^2}$$

$$= \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + j2\xi\left(\frac{\omega}{\omega_n}\right) + 1}$$

Let $u = \frac{\omega}{\omega_n}$,

(2) becomes :-

$$M(ju) = \frac{1}{1 + j2\xi u - u^2} \quad \text{— (3)}$$

The magnitude and phase of (3) are:

$$|M(ju)| = \frac{1}{\left[(1-u^2)^2 + (2\xi u)^2\right]^{1/2}}$$

and $\angle M(ju) = \phi_M(ju) = -\tan^{-1}\left[\frac{2\xi u}{1-u^2}\right]$

The resonant frequency of system is determined by setting $\frac{d|M(ju)|}{du} = 0$

$$\Rightarrow \frac{d}{du}\left[\frac{1}{\left[(1-u^2)^2 + (2\xi u)^2\right]^{1/2}}\right] = 0$$

$$\Rightarrow -\frac{1}{2}\left[(1-u^2)^2 + (2\xi u)^2\right]^{-3/2}\left[2(1-u^2)(0-2u) + 2(2\xi u)(2\xi)\right] = 0$$

$$\Rightarrow \left[4u^3 - 4u + 8\xi^2 u\right] = 0$$

$$\Rightarrow 4u\left[u^2 - 1 + 2\xi^2\right] = 0$$

$$\therefore \quad u = 0 \quad \text{or} \quad u = \sqrt{1 - 2\xi^2}$$

Normalized resonant frequency is :- $\frac{\omega_r}{\omega_n} = u_r = \sqrt{1-2\xi^2}$

and the resonant frequency is :- $\omega_r = \omega_n\sqrt{1-2\xi^2}$.

Since frequency is a real quantity,

$$2\xi^2 \le 1$$

$$\Rightarrow \xi < \frac{1}{\sqrt{2}}$$

or $\boxed{\xi \le 0.707}$

For all values of $\xi > 0.707$, $\boxed{\omega_r = 0 \text{ and } M_r = 1}$

$\therefore$ Resonant peak $M_r = \left| M(j\omega) \right|_{u = u_r = \sqrt{1-2\xi^2}}$

$$= \frac{1}{\left[(1-u_r^2)^2 + (2\xi u_r)^2 \right]^{1/2}}$$

$$= \frac{1}{\left[(1-1+2\xi^2)^2 + 4\xi^2(1-2\xi^2) \right]^{1/2}}$$

$$= \frac{1}{\left[4\xi^4 + 4\xi^2 - 8\xi^4 \right]^{1/2}}$$

$$= \frac{1}{\left[4\xi^2 - 4\xi^4 \right]^{1/2}}$$

$$= \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\therefore \boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}, \quad \xi \le 0.707$$

Band Width (BW)

In accordance with the definition of bandwidth,

$$|M(ju)| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\left[(1-u^2)^2+(2\xi u)^2\right]^{1/2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \left[(1-u^2)^2+(2\xi u)^2\right]^{1/2} = \sqrt{2}$$

$$\Rightarrow (1-u^2)^2+(2\xi u)^2 = 2$$

$$\Rightarrow 1+u^4-2u^2+4\xi^2u^2-2=0$$

$$\Rightarrow u^4+(4\xi^2-2)u^2-1=0$$

Let $u^2=m$

$$\therefore m^2+(4\xi^2-2)m-1=0$$

$$\therefore m = \frac{-(4\xi^2-2)\pm\sqrt{(4\xi^2-2)^2+4}}{2}$$

$$= (1-2\xi^2)\pm\sqrt{4\xi^4-4\xi^2+2}$$

Considering the sign.

$$u = \sqrt{(1-2\xi^2)+\sqrt{(1-2\xi^2)^2+1}}$$

$\therefore$ Band width

$$\boxed{BW = \omega_n\sqrt{(1-2\xi^2)+\sqrt{(1-2\xi^2)^2+1}}}$$

Correlation between time-domain & Frequency Domain specifications :-

$$r(t) \longrightarrow \boxed{\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}} \longrightarrow y(t)$$

Prototype Second Order System.

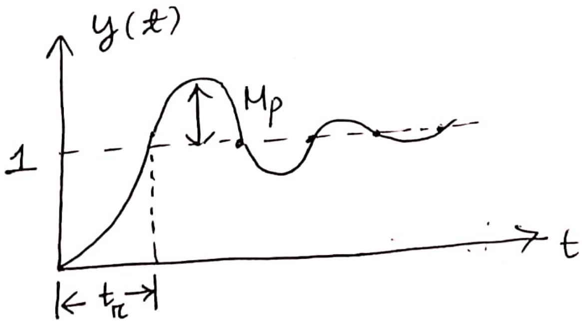
Time Domain Specifications

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}, \quad 0<\xi<1$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$t_r = \frac{\pi-\phi}{\omega_d}$$

$$= \frac{\pi-\cos^{-1}(\xi)}{\omega_n\sqrt{1-\xi^2}}$$



$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\phi), \quad t\geq 0$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

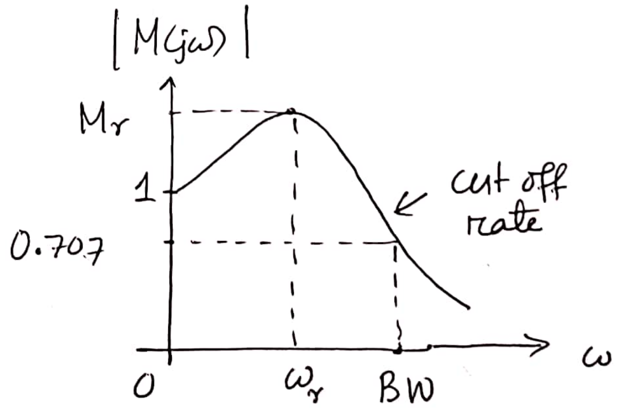
$$\cos\phi = \xi$$

Frequency Domain Specifications

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \qquad \xi \leq 0.707$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$BW = \omega_n\sqrt{(1-2\xi^2)+\sqrt{(1-2\xi^2)^2+1}}$$



→ As ω_n gets larger, t_r gets smaller and the system responds faster.

→ As ξ gets larger, t_r gets larger and the system responds slower.

→ As ω_n gets larger, BW gets larger.

→ As ξ gets larger, BW gets smaller.

Finally,

→ Bandwidth and risetime are inversely proportional

→ Increasing ω_n, increases BW and decreases t_r

→ Increasing ξ, decreases BW and increases t_r.

<u>Questions</u> :-

Q. (1) Derive the expressions for following frequency domain specifications for a proto-type second order system :-

i) Resonant Frequency

ii) Resonant Peak.

iii) Band width

Q. (2) Write down the correlation between time-domain and frequency domain specifications for a proto-type second order system.

4.2 POLAR PLOT

→ Two methods of determining stability of linear SISO system : Routh-Hurwitz Criteria, Root-Locus method. → based on locating the roots of characteristics equation in the s-plane.

→ The Nyquist Criterion is a semi-graphical method that determines the stability of a closed loop system by investigating the properties of the frequency domain plot, the Nyquist plot, of the open-loop transfer function $G(s)H(s)$.

→ Nyquist plot is plot $G(j\omega)H(j\omega)$ in the polar coordinates of $Im[G(j\omega)H(j\omega)]$ Vs. $Re[G(j\omega)H(j\omega)]$ as ω varies from 0 to ∞. That is why the Nyquist plot as ω varies from 0 to ∞ is known as Polar plot.

Features of Polar Plot :-

① Polar plot/Nyquist Plot gives information on the relative stability of the stable system, and the degree of instability of an unstable system. It gives indication on how system stability may be improved.

② Polar plot/Nyquist Plot is very easy to obtain, especially with the aid of a computer.

③ The Polar plot of $G(s)H(s)$ gives information on the frequency domain characteristics such as M_r, ω_r, BW

④ Nyquist/Polar plot is useful for systems with pure time delay. that cannot be treated with the Routh-Hurwitz Criterion and are difficult to analyze with Root-Locus method.

(5) Unlike the Root-locus method Nyquist criterion doesnot give the exact location of the characteristics equation roots.

Let us consider the CLTF of a SISO system,

$$M(s) = \frac{G(s)}{1+G(s)H(s)}.$$

Where $G(s)H(s)$: OLTF can assume the following form:

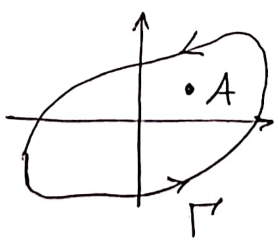
$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots(1+T_m s)}{s^p(1+T_a s)(1+T_b s)\cdots(1+T_n s)}\, e^{-T_d s}$$

Where the T's are real or complex conjugate coefficients and T_d is a real time delay.

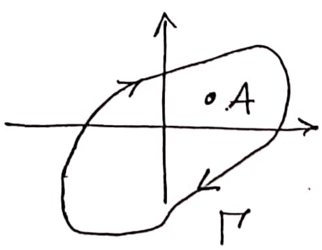
Roots of the characteristics equation are also zeros of $1+G(s)H(s)=0$

$\therefore$ Closed loop transfer Function poles $\triangleq$ Zeros of $1+G(s)H(s)=0$
$\triangleq$ Roots of Characteristics Equation.

Concept of Enclosement :- [Convention used here]



Point A is not enclosed by contour Γ



Point A is enclosed by contour Γ

A point is said to be enclosed by a contour or closed path if it is to the right hand side of direction of contour.

Critical Point :

Characteristics equation,

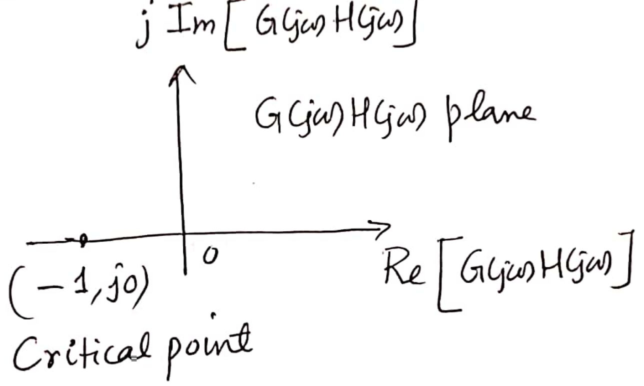
$$1 + G(s)H(s) = 0$$

$$\Rightarrow G(s)H(s) = -1$$

Put $s = j\omega$

$$G(j\omega)H(j\omega) = -1 = -1 + j0$$

The critical point is $(-1, j0)$ in the $G(j\omega)H(j\omega)$ plane.



Closed loop Stability From Polar Plot

A closed loop system is said to be absolute stable if the polar plot doesnot enclose the critical point (-1, j0)
If the plot encloses the critical point, the ~~plot bee~~ closed loop system becomes unstable.

Steps to be followed to determine closed loop System stability from polar plot

(i) Draw the polar plot
(ii) Determine the point of intersection of polar plot with Real axis
(iii) Locate the critical point $(-1, j0)$
(iv) Check if the critical point is enclosed by polar plot.
(v) not enclosed → Close loop stable; enclosed – close loop unstable

Q) Determine the range of values of K for which the system having OLTF

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$ is stable.

Soln:- Given OLTF :

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

$$= \frac{K}{2s(1+s)(1+\frac{1}{2}s)}$$

$$= \frac{\left(\frac{K}{2}\right)}{s(1+s)(1+\frac{1}{2}s)} \quad \text{(Time constant Form)}$$

$$= \frac{K_1}{s(1+s)(1+\frac{1}{2}s)} \quad \left(\because K_1 = \frac{K}{2}\right)$$

$$= K_1\, G_1(s)H_1(s)$$

Let $G_1(s)H_1(s) = \dfrac{1}{s(1+s)(1+\frac{1}{2}s)}$

We draw the polar plot of $G_1(s)H_1(s)$

Put $s = j\omega$

$$G_1(j\omega)H_1(j\omega) = \frac{1}{j\omega(1+j\omega)(1+\frac{1}{2}j\omega)} \quad \text{—(1)}$$

$$|G_1(j\omega)H_1(j\omega)| = \left|\frac{1}{j\omega(1+j\omega)(1+\frac{1}{2}j\omega)}\right| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+\frac{1}{4}\omega^2)}}$$

$$\angle G_1(j\omega)H_1(j\omega) = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\tfrac{1}{2}\omega\right)$$

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ω	$\lvert G_1(j\omega)H_1(j\omega)\rvert = \frac{1}{\omega\sqrt{(1+\omega^2)(1+\omega^2/4)}}$	$\angle G_1(j\omega)H_1(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}\left(\frac{\omega}{2}\right)$
0	∞	-90°
∞	0	-270°

ω	$Re[G_1(j\omega)H_1(j\omega)] = \frac{-3/2}{(1+\omega^2)(1+\omega^2/4)}$	$Im[G_1(j\omega)H_1(j\omega)] = \frac{-(1-\omega^2/2)}{\omega(1+\omega^2)(1+\omega^2/4)}$
0	$-3/2$	$-\infty$
∞	-0	$+0$

From ① $G_1(j\omega)H_1(j\omega) = \frac{(1+j0)}{(0+j\omega)(1+j\omega)(1+\frac{1}{2}j\omega)} = \frac{(0-j\omega)(1-j\omega)(1-\frac{1}{2}j\omega)}{\omega^2(1+\omega^2)(1+\frac{\omega^2}{4})}$

$$= \frac{(-j\cancel{\omega})(1-j\frac{3}{2}\omega-\frac{\omega^2}{2})}{\omega^{\cancel{2}}(1+\omega^2)(1+\omega^2/4)}$$

$$= \frac{-3/2\,\omega - j(1-\frac{\omega^2}{2})}{\omega(1+\omega^2)(1+\omega^2/4)}$$

$$= \frac{-\frac{3}{2}\cancel{\omega}}{\cancel{\omega}(1+\omega^2)(1+\frac{\omega^2}{4})} - j\frac{(1-\frac{\omega^2}{2})}{\omega(1+\omega^2)(1+\frac{\omega^2}{4})}$$

$\therefore$ $Re[G_1(j\omega)H_1(j\omega)] = \frac{-3/2}{(1+\omega^2)(1+\frac{\omega^2}{4})}$

$Im[G_1(j\omega)H_1(j\omega)] = \frac{-(1-\frac{\omega^2}{2})}{\omega(1+\omega^2)(1+\frac{\omega^2}{4})}$

On Real axis $Im[G_1(j\omega)H_1(j\omega)] = 0$

$$\Rightarrow 1-\frac{\omega^2}{2} = 0$$

$$\Rightarrow \frac{\omega^2}{2} = 1 \Rightarrow \omega^2 = 2 \Rightarrow \boxed{\omega = \sqrt{2}}$$

And $Re[G_1(j\omega)H_1(j\omega)]\Big|_{\omega=\sqrt{2}} = \frac{-3/2}{(1+2)(1+\frac{2}{4})} = -\frac{1}{3}$

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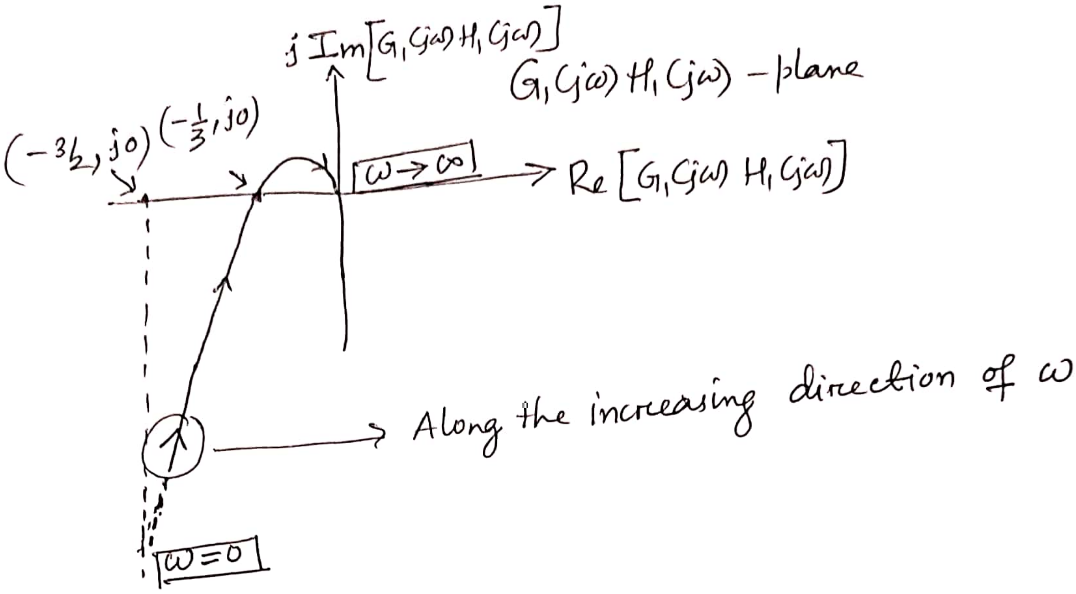
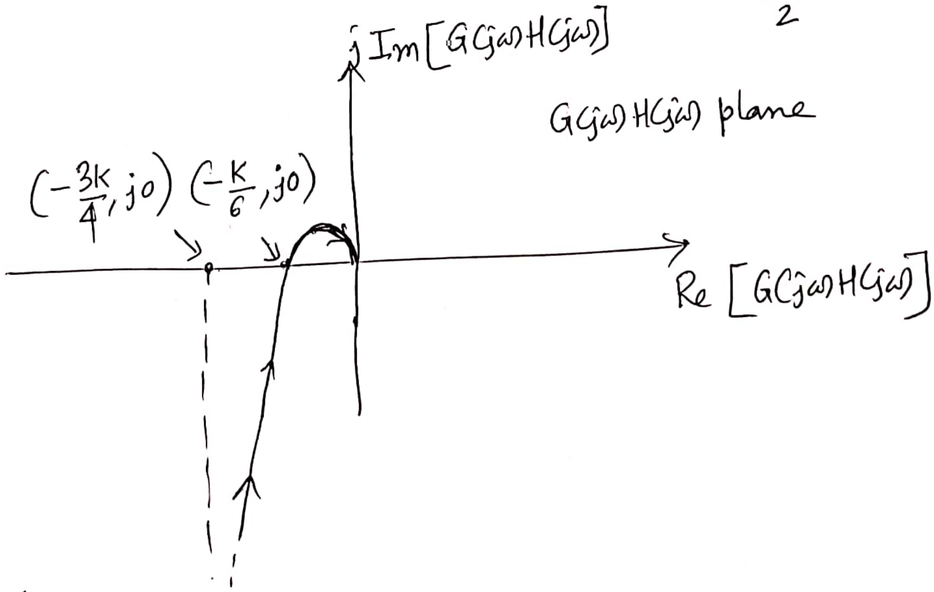


Polar plot of $G(s)H(s) = K_1 G_1(s)H_1(s)$

$$= \frac{K}{2} G_1(s)H_1(s)$$



Closed loop Stability of System

Absolute Stable	Marginally Stable	Unstable
→ Critical point shouldnot be enclosed	→ Critical point should lie on the plot	→ Critical point should be enclosed
→ (−1, j0) should lie to the LHS of plot	→ (−1, j0) should lie on the plot	→ (−1, j0) should lie to the RHS of plot
$-\frac{K}{6} > -1$	$-\frac{K}{6} = -1$	$-\frac{K}{6} < -1$
$\Rightarrow \frac{K}{6} < 1$	$\Rightarrow K = 6$	$\Rightarrow \frac{K}{6} > 1$
$\Rightarrow \boxed{K < 6}$	$\Rightarrow \boxed{K = 6}$	$\Rightarrow \boxed{K > 6}$

Q. Plot the Polar plot of system having open loop transfer function $G(s)H(s) = \frac{K}{s(1+T_1 s)(1+T_2 s)}$ and determine its stability using concept of enclosement.

Soln:- Given OLTF of system

$$G(s)H(s) = \frac{K}{s(1+T_1 s)(1+T_2 s)}$$

$$= K\, G_1(s)\, H_1(s)$$

Let us draw the polar plot of $G_1(s)H_1(s)$.

$$\therefore\ G_1(s)H_1(s) = \frac{1}{s(1+T_1 s)(1+T_2 s)} \quad \text{(time constant form)}$$

Put $s = j\omega$

$$G_1(j\omega)H_1(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$|G_1(j\omega)H_1(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}$$

$$\angle G_1(j\omega)H_1(j\omega) = -90^\circ - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Also: $$G_1(j\omega)H_1(j\omega) = \frac{1(0-j\omega)(1-j\omega T_1)(1-j\omega T_2)}{(0^2+\omega^2)(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$= \frac{(-j\omega)(1-j\omega T_1-j\omega T_2-\omega^2T_1T_2)}{\omega^2(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$= \frac{-\omega(T_1+T_2) - j(1-\omega^2T_1T_2)}{\omega(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$= \frac{-\omega(T_1+T_2)}{\omega(1+\omega^2T_1^2)(1+\omega^2T_2^2)} - j\frac{(1-\omega^2T_1T_2)}{\omega(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$\therefore \; Re\left[G_1(j\omega)H_1(j\omega)\right] = \frac{-\cancel{\omega}(T_1+T_2)}{\cancel{\omega}(1+\omega^2T_1^2)(1+\omega^2T_2^2)} = \frac{-(T_1+T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$Im\left[G_1(j\omega)H_1(j\omega)\right] = \frac{-(1-\omega^2T_1T_2)}{\omega(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

ω	$\lvert G_1(j\omega)H_1(j\omega)\rvert = \frac{1}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}$	$\angle G_1(j\omega)H_1(j\omega) = -90^\circ - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$	$Re[G_1(j\omega)H_1(j\omega)] = \frac{-(T_1+T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$	$Im[G_1(j\omega)H_1(j\omega)] = \frac{-(1-\omega^2T_1T_2)}{\omega(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$
0	∞	-90°	$-(T_1+T_2)$	$-\infty$
∞	0	-270°	-0	$+0$

Intersection of Polar plot with Real axis.

$$Im\left[G_1(j\omega)H_1(j\omega)\right] = 0$$

$$\Rightarrow \quad 1-\omega^2T_1T_2 = 0$$

$$\Rightarrow \quad \omega^2T_1T_2 = 1$$

$$\Rightarrow \quad \boxed{\omega = \frac{1}{\sqrt{T_1T_2}}}$$

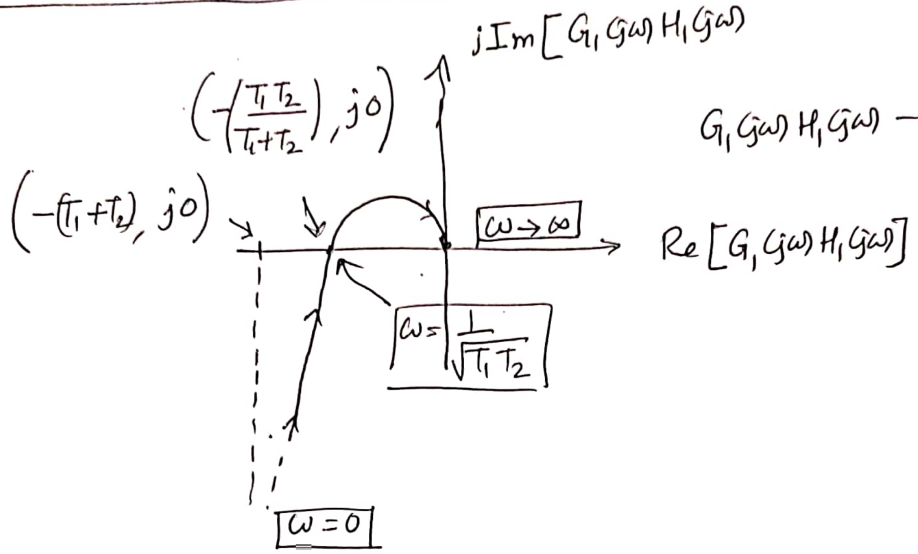
$$Re\left[G_1(j\omega)H_1(j\omega)\right]\Big|_{\omega=\frac{1}{\sqrt{T_1T_2}}}$$

$$= \frac{-(T_1+T_2)}{\left(1+\frac{T_1^2}{T_1T_2}\right)\left(1+\frac{T_2^2}{T_1T_2}\right)}$$

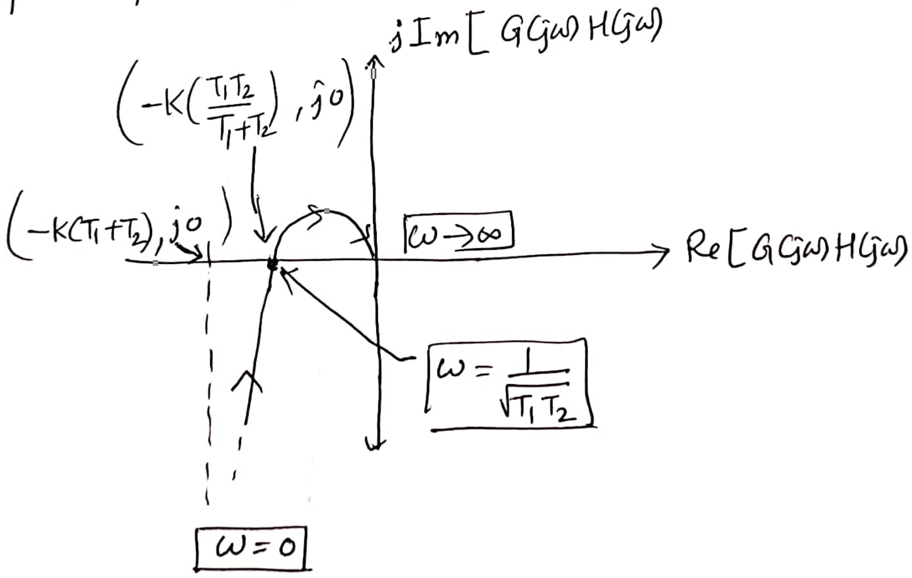
$$= \frac{-(T_1+T_2)}{\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)}$$

$$= \frac{-(T_1+T_2)}{\frac{(T_1+T_2)^2}{T_1T_2}} = -\left(\frac{T_1T_2}{T_1+T_2}\right)$$

Polar plot of $G_1(s)H_1(s)$



Polar plot of $G(s)H(s) = K\,G_1(s)H_1(s)$



Closed loop stability of System :-

Absolute Stable

→ Critical point shouldn't be enclosed

→ Critical point $(-1, j0)$ should lie to the LHS of plot

$$-K\left(\frac{T_1T_2}{T_1+T_2}\right) > -1$$

$$\Rightarrow \boxed{K < \frac{T_1+T_2}{T_1T_2}}$$

Marginally Stable

→ Critical point should lie on the plot

→ $(-1, j0)$ should lie on the plot

$$-\left(\frac{KT_1T_2}{T_1+T_2}\right) = -1$$

$$\Rightarrow \boxed{K = \frac{T_1+T_2}{T_1T_2}}$$

Unstable

→ Critical point should be enclosed

→ $(-1, j0)$ should lie to the RHS of plot

$$-K\left(\frac{T_1T_2}{T_1+T_2}\right) < -1$$

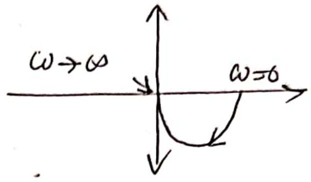
$$\Rightarrow \boxed{K > \frac{T_1+T_2}{T_1T_2}}$$

Some standard Polar plots :-

OLTF	Polar plot

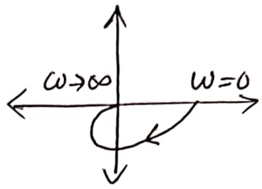
1. $G(s)H(s) = \dfrac{1}{(1+Ts)}$

   Type → 0, Order → 1



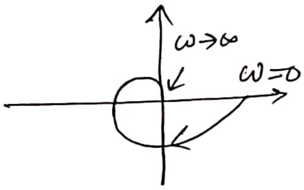
2. $G(s)H(s) = \dfrac{1}{(1+T_1 s)(1+T_2 s)}$

   Type → 0, Order → 2



3. $G(s)H(s) = \dfrac{1}{(1+T_1 s)(1+T_2 s)(1+T_3 s)}$

   Type → 0, Order → 3



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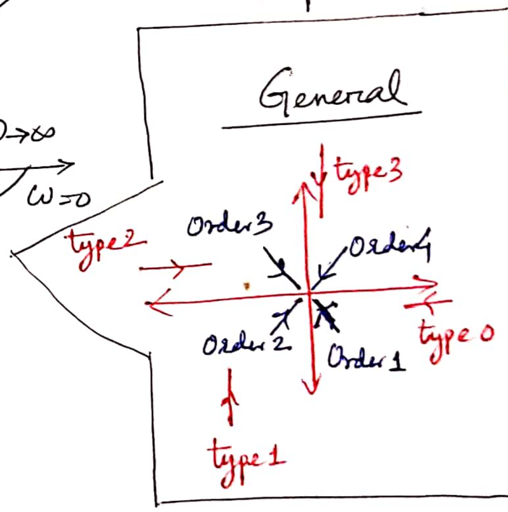
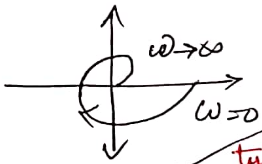


4. $G(s)H(s) = \dfrac{1}{(1+T_1 s)(1+T_2 s)(1+T_3 s)(1+T_4 s)}$

   Type → 0, Order → 4



5. $G(s)H(s) = \dfrac{1}{s}$

   Type → 1, Order − 1

6. $G(s)H(s) = \dfrac{1}{s(1+Ts)}$

   Type-1, Order:2

7. $G(s)H(s) = \dfrac{1}{s(1+T_1 s)(1+T_2 s)}$

   Type-1, Order → 3